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Occupation times for planar and higher dimensional Brownian motion

Jean Desbois

CNRS; University Paris Sud, UMR8626, LPTMS, ORSAY CEDEX, F-91405, France

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Abstract
We consider a planar Brownian motion starting from O at time $t = 0$ and stopped at t. Denoting by T the time spent in a wedge of apex O and angle Θ, we develop a method to compute systematically the moments of T for general Θ values. We apply it to obtain analytically the second and third moments for a general wedge angle and, also, the fourth moment for the quadrant ($\Theta = \pi/2$). We compare our results with numerical simulations. Finally, with standard perturbation theory, we establish a general formula for the second moment of an orthant occupation time.

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1. Introduction

The study of occupation times for Brownian motion (BM) traces back to the famous Levy's arc-sine law [1, 2]. It states that, for a one-dimensional Brownian motion ($BM(\mathbf{R})$) starting at the origin at $t = 0$ and stopped at t, the time T spent when $x > 0$ satisfies the probability law ($0 < t_1 < t$):

$$P(T < t_1) = \frac{2}{\pi} \arcsin \sqrt{\frac{t_1}{t}} \tag{1}$$

with the density

$$\mathcal{P}(T) = \frac{1}{\pi} \frac{1}{\sqrt{T(t-T)}} \tag{2}$$

and the Laplace transform ($\gamma > 0, \gamma + \xi > 0$):

$$\int_0^\infty \mathrm{d}t \, \mathrm{e}^{-\gamma t} \langle \mathrm{e}^{-\xi T} \rangle_{\mathcal{C}_t} = \frac{1}{\sqrt{\gamma(\gamma + \xi)}}, \tag{3}$$

where $\langle \ldots \rangle_{\mathcal{C}_t}$ stands for averaging over all Brownian curves of length t starting from O.

Barlow, Pitman and Yor [3] have extended Levy's result to a BM on a set of n semi-infinite lines originating from O (a kind of special graph). $\{T_i\}$ being the set of occupation times on all the lines, the authors computed explicitly the Laplace transform of the $\{T_i\}$ distribution. This study has again been extended more recently [4] to a BM on a quite general graph. This time, the Laplace transform appears in a closed form as the ratio of two determinants. At that moment, we could think that a way is opened to solve the occupation time problem for a general n-dimensional Brownian motion ($BM(\mathbf{R}^n)$). For instance, it is possible, for a planar BM, to formulate the long-standing quadrant occupation time problem by using a 2D square lattice. Unfortunately, with [4], it is very difficult to extract clear information on the occupation time distribution for such an infinite graph.

On the other hand, Bingham and Doney [5] computed the first moments of the quadrant occupation time T for the planar BM. Taking advantage of the independence of the coordinate processes, they got

$$\left\langle \left(\frac{T}{t}\right)^2 \right\rangle = \frac{5}{32} - \frac{1}{8\pi^2} = 0.143\,58\ldots \tag{4}$$

$$\left\langle \left(\frac{T}{t}\right)^3 \right\rangle = \frac{53}{512} + \frac{7}{18\pi} - \frac{347}{288\pi^2} = 0.105\,22\ldots. \tag{5}$$

Moreover, they argued that the computation of the higher moments seems to be very difficult.

Nevertheless, they concluded from their study that the law of T is neither a generalized arc-sine nor a beta distribution (see equation (7) for the definition). The approach we develop in the following will confirm their conclusion despite the fact that we disagree with their value for the third moment. We will come back to this point later.

In their analysis of the occupation times of cones, Meyre and Werner [6] establish several theorems concerning the asymptotic behaviour of T. At the end of their paper, the authors put forward some conjectures that come out quite naturally from their work. In particular, for planar BM, they expect the following expression for the occupation time density of a wedge of apex O and magnitude Θ:

$$\mathcal{P}_C\left(\frac{T}{t} = u\right) = h_C(u)u^{\frac{\pi}{2(2\pi-\Theta)}-1}(1-u)^{\frac{\pi}{2\Theta}-1}, \tag{6}$$

where $h_C(u)$ is a bounded continuous and strictly positive function on [0, 1].

A constant h_C function would correspond to a beta distribution[1]:

$$B_{\alpha,\beta}(u) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}(1-u)^{\alpha-1}u^{\beta-1}, \tag{7}$$

with $\alpha = \frac{\pi}{2\Theta}$ and $\beta = \frac{\pi}{2(2\pi-\Theta)}$. Recall that the moments of the density (7) are given by

$$\langle u^k\rangle = \frac{\beta(\beta+1)\cdots(\beta+k-1)}{(\alpha+\beta)(\alpha+\beta+1)\cdots(\alpha+\beta+k-1)}. \tag{8}$$

Focusing on the conjecture equation (6), for the quadrant ($\Theta = \pi/2$), it leads to

$$\mathcal{P}_C(u) = h_C(u)u^{-\frac{2}{3}}. \tag{9}$$

Actually, we have checked by numerical simulations that $h_C(u)$ is close to a constant ($= 1/3$) for all $u \in [0, 1]$. Thus, we are convinced that equation (6) captures the essential features of the density that is quite close to a beta law.

[1] This is actually what happens for $\Theta = \pi$ (half-plane occupation time problem) where Levy's arc-sine law still holds. This fact is due to a factorization property of the planar BM (the component of the BM parallel to the frontier of the half-plane plays no role; so, the problem narrows down to the one-dimensional one: the occupation time density is given by equation (2) or, equivalently, by the distribution $B_{1/2,1/2}(u)$, equation (7)).

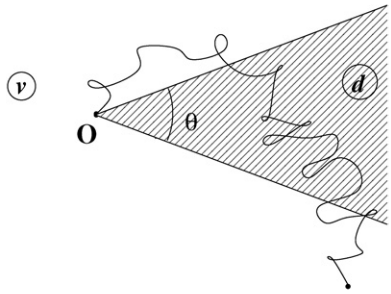

Figure 1. The Brownian particle starts from O at $t = 0$ and stops somewhere in the plane at time t. It spends a time T in the wedge d.

In this paper, we propose to develop, for the planar BM, a systematic method to compute the moments of T for a general wedge. In particular, we do not restrict our study to the quadrant problem as was done in [5] (no coordinate processes independence at hand). We will also briefly address the orthant occupation time problem for a $BM(\mathbf{R}^n)$. In that case, we will establish, for a general n, an explicit formula for the second moment of T.

2. Planar Brownian motion

Let us consider a Brownian particle (see figure 1) starting from O at $t = 0$ and stopping somewhere in the plane at time t. We call T the time spent in the sector d of apex O and angle $\Theta \equiv 2\alpha$. Our goal is to compute the moments of the random variable T.

Due to the scaling property of the BM, it is convenient to introduce the reduced variable $u \equiv \frac{T}{t}$ ($u \in [0, 1]$) and to consider the quantity ($1 + p > 0$):

$$L = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t} \langle \mathrm{e}^{-pT} \rangle_{\mathcal{C}_t} = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t} \langle \mathrm{e}^{-put} \rangle_{\mathcal{C}} = \left\langle \frac{1}{1 + pu} \right\rangle_{\mathcal{C}} \tag{10}$$

where $\langle \ldots \rangle_{\mathcal{C}}$ stands for averaging over all Brownian curves starting from O. (Recall that $\langle \ldots \rangle_{\mathcal{C}_t}$ is the same but for Brownian curves of length t.)

Alternatively, $\langle \mathrm{e}^{-pT} \rangle_{\mathcal{C}_t}$ can be written in terms of path integrals [7]:

$$\langle \mathrm{e}^{-pT} \rangle_{\mathcal{C}_t} = \int_{\mathbf{R}^2} \mathrm{d}\vec{R} \int_{\vec{r}(0)=\vec{0}}^{\vec{r}(t)=\vec{R}} \mathcal{D}\vec{r}(\tau) \exp\left(- \int_0^t \left(\frac{\dot{\vec{r}}(\tau)^2}{4D} + p\mathbf{1}_d(\vec{r}(\tau)) \right) \mathrm{d}\tau \right) \tag{11}$$

$$= \int_{\mathbf{R}^2} \mathrm{d}\vec{R} \langle \vec{R} | \mathrm{e}^{-Ht} | \vec{0} \rangle \tag{12}$$

with

$$H = -D\Delta + p\mathbf{1}_d(\vec{r}). \tag{13}$$

In equations (11) and (12), we integrate over the final point $\vec{R}$ because this point is left undetermined. $\mathbf{1}_d(\vec{r})$ is the characteristic function of the domain d ($\mathbf{1}_d(\overrightarrow{OM}) = 1$ if $M \in d$, $= 0$ otherwise). D is the diffusion constant. Owing to scaling properties, the value of D is irrelevant for the problem at hand (this will be explicitly shown in the following section). Here, we set $D = 1$.

Equations (10) and (12) lead to

$$L = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t} \int_{\mathbf{R}^2} \mathrm{d}\vec{R} \langle \vec{R} | \mathrm{e}^{-Ht} | \vec{0} \rangle = \int_{\mathbf{R}^2} \mathrm{d}\vec{R} \langle \vec{R} | \frac{1}{H + 1} | \vec{0} \rangle. \tag{14}$$

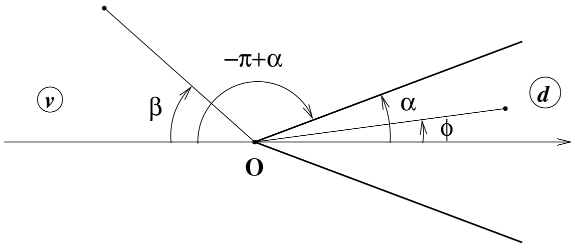

Figure 2. The definition of the angles used in ψ_d and ψ_v.

Equations (13) and (14) show that we can write L as $L = \psi\ (\vec{r} = \vec{0})$, where the function ψ ($\equiv \psi_d(r,\phi)$ on d, $\equiv \psi_v(r,\beta)$ on v) satisfies the following backward equation ($n^2 \equiv 1 + p$):

$$(-\Delta + n^2)\psi_d = 1 \tag{15}$$

$$(-\Delta + 1)\psi_v = 1 \tag{16}$$

Continuity of ψ and of its derivative lead to the boundary conditions (see figure 2 for the notations)

$$\psi_d(r, \phi = \alpha) = \psi_v(r, \beta = -\pi + \alpha) \tag{17}$$

$$\frac{\partial \psi_d}{\partial \phi}(r, \phi = \alpha) = \frac{\partial \psi_v}{\partial \beta}(r, \beta = -\pi + \alpha) \tag{18}$$

Owing to symmetry properties, the general solution of equations (15) and (16) can be written in terms of two arbitrary functions, $a(\nu)$ and $b(\nu)$ [8]:

$$\psi_d(r,\phi) = \frac{2}{\pi}\int_0^{\infty} \mathrm{d}\nu\, a(\nu)\cosh(\nu\phi)K_{i\nu}(nr) + \frac{1}{n^2} \tag{19}$$

$$\psi_v(r,\beta) = \frac{2}{\pi}\int_0^{\infty} \mathrm{d}\nu\, b(\nu)\cosh(\nu\beta)K_{i\nu}(r) + 1. \tag{20}$$

$K_{i\nu}$ is a modified Bessel function of an imaginary index [9, 10]. Recall that

$$K_{i\nu}(r) = \frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{e}^{-r\cosh x}\cos\nu x\,\mathrm{d}x \to_{r\to 0} \pi\delta(\nu). \tag{21}$$

Then, conditions (17) and (18) lead to a set of two coupled integral equations for $a(\nu)$ and $b(\nu)$.

In order to get rid of $b(\nu)$, we will use the relationship [8], $\left(U \equiv 1 - \frac{1}{n^2} = \frac{p}{1+p}\right)$:

$$\int_0^{\infty} \frac{\mathrm{d}r}{r} K_{i\nu}(nr)K_{i\sigma}(r) = \frac{\pi^2}{2\sigma}\frac{\cos(\sigma\ln n)}{\sinh(\sigma\pi)}\delta(\sigma-\nu) + U\frac{\pi^2}{4}n^{-i\sigma}\frac{F}{\cosh(\pi\nu)-\cosh(\pi\sigma)} \tag{22}$$

$$F \equiv F\left(1 + \frac{\mathrm{i}}{2}(\sigma+\nu); 1 + \frac{\mathrm{i}}{2}(\sigma-\nu); 2; U\right), \tag{23}$$

where F is an hypergeometric function [10] and the last term in (22) has to be understood as the principal value (of course, the same prescription must hold for all the integrals to be computed in the following).

Finally, we are left with the following integral equation for the function $a(\sigma)$:

$$2a(\sigma)\cos(\sigma\ln n) + U(1-U)^{i\sigma/2}\mathcal{O}(a(\nu)F) = \frac{U\sinh(\sigma(\pi-\alpha))}{\sinh(\sigma\pi/2)}, \tag{24}$$

where F is defined in equation (23) and

$$\mathcal{O}(G) = \sigma \sinh(\sigma(\pi - \alpha)) \int_0^\infty \mathrm{d}\nu \frac{\cosh(\nu\alpha)G}{\cosh(\pi\nu) - \cosh(\pi\sigma)} + \cosh(\sigma(\pi - \alpha)) \int_0^\infty \mathrm{d}\nu \frac{\nu \sinh(\nu\alpha)G}{\cosh(\pi\nu) - \cosh(\pi\sigma)}. \tag{25}$$

With the U-expansions

$$a(\nu) = \sum_{j=0}^{\infty} a_j(\nu) U^j \tag{26}$$

$$(1 - U)^{i\sigma/2} F = \sum_{j=0}^{\infty} c_j(\sigma, \nu) U^j \tag{27}$$

$$\cos(\sigma \ln n) = 1 - \frac{\sigma^2 U^2}{8} - \frac{\sigma^2 U^3}{8} + \cdots, \tag{28}$$

equation (24) leads to the following recursion relations:

$$a_0(\sigma) = 0 \tag{29}$$

$$2a_1(\sigma) = \frac{\sinh(\sigma(\pi - \alpha))}{\sinh(\sigma\pi/2)} \tag{30}$$

$$2a_2(\sigma) = -\mathcal{O}(a_1(\nu)) \tag{31}$$

$$2a_3(\sigma) = \frac{\sigma^2}{4} a_1(\sigma) - \mathcal{O}(a_1(\nu)c_1(\sigma, \nu) + a_2(\nu)) \tag{32}$$

$$2a_4(\sigma) = \frac{\sigma^2}{4}(a_2(\sigma) + a_1(\sigma)) - \mathcal{O}(a_1(\nu)c_2(\sigma, \nu) + a_2(\nu)c_1(\sigma, \nu) + a_3(\nu)) \tag{33}$$

$$\ldots.$$

It is now easy to come back to the moments of u.

With equations (19) and (21), we get $L = \psi_d(\vec{0}) = a(0) + 1 - U$. Thus, expanding L in equation (10), we obtain

$$\langle u \rangle = 1 - a_1(0) \tag{34}$$

$$\langle u^2 \rangle = \langle u \rangle + a_2(0) \tag{35}$$

$$\langle u^3 \rangle = 2\langle u^2 \rangle - \langle u \rangle - a_3(0) \tag{36}$$

$$\langle u^4 \rangle = 3\langle u^3 \rangle - 3\langle u^2 \rangle + \langle u \rangle + a_4(0) \tag{37}$$

$$\ldots.$$

Equations (30) and (34) give, as expected:

$$\langle u \rangle = \frac{\alpha}{\pi} = \frac{\Theta}{2\pi}. \tag{38}$$

Now, in view of the second moment $\langle u^2 \rangle$, let us sketch the computation of $a_2(\sigma)$. Equation (31) can be written as

$$2a_2(\sigma) = -\sigma \sinh(\sigma(\pi - \alpha))K(\alpha) - \cosh(\sigma(\pi - \alpha))\frac{\mathrm{d}K}{\mathrm{d}\psi}(\psi = \alpha) \tag{39}$$

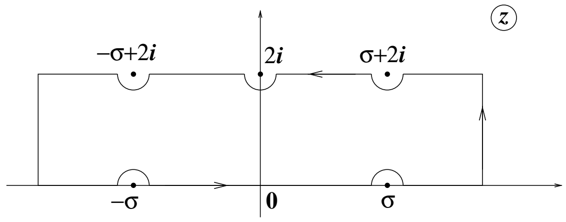

Figure 3. The contour, in the complex z-plane, that was used to compute $N(\phi)$, equations (41) and (42).

$$K(\psi) = \frac{1}{4}(N(\pi - \alpha + \psi) + N(\pi - \alpha - \psi)) \tag{40}$$

$$N(\phi) = \int_0^\infty \mathrm{d}\nu \frac{\sinh(\nu\phi)}{\sinh(\nu\pi/2)(\cosh(\nu\pi) - \cosh(\sigma\pi))} \tag{41}$$

$$= \tan\phi\left(\frac{\cosh(\sigma\phi) - \cosh(\sigma\pi/2)}{\sinh(\sigma\pi/2)\sinh(\sigma\pi)}\right), \tag{42}$$

where $N(\phi)$ has been obtained by the contour integration of the function

$$f(z) = \frac{\sinh(z\phi)}{\sinh(z\pi/2)(\cosh(z\pi) - \cosh(\sigma\pi))}$$

(see figure 3 for the contour).

The result for $a_2(\sigma)$ writes

$$a_2(\sigma) = -\frac{1}{8\sinh(\sigma\pi/2)\sinh(\sigma\pi)}\bigg[\sigma\tan(2\alpha)(-\sinh(\sigma\alpha) + \sinh(\sigma(\pi-\alpha))\cosh(\sigma\pi/2)) + \cosh(\sigma(\pi-\alpha))\bigg(\cosh(\sigma\pi) - \cosh(\sigma\pi/2) - \frac{1}{\cos^2 2\alpha}(\cosh(\sigma(\pi - 2\alpha)) - \cosh(\sigma\pi/2))\bigg)\bigg]. \tag{43}$$

With (35), we get for the second moment of the occupation time of a wedge of angle Θ:

$$\langle u^2\rangle_{(\Theta)} = \frac{\Theta}{2\pi} + \frac{1}{8\pi^2}\left[\left(\left(\frac{\pi}{2} - \Theta\right)\tan\Theta - 1\right)\left(\left(\frac{3\pi}{2} - \Theta\right)\tan\Theta - 1\right) - \Theta(2\pi - \Theta) - 1\right]. \tag{44}$$

Remark that the relationship $\langle u^n\rangle_{(2\pi-\Theta)} = \langle(1-u)^n\rangle_{(\Theta)}$ should hold for all n values. Here, with (38) and (44), it is easy to check it for $n = 2$.

Moreover, with (44), it is possible to show that, for all the Θ values, the density of u cannot be a beta law with the asymptotic behaviour conjectured by Meyre and Werner [6] (except, of course, for $\Theta = \pi$ where Levy's arc-sine law holds). Indeed, with the beta law equation (7), we should get, for the second moment

$$\langle u^2\rangle_{(\Theta)} = \frac{\Theta^2\left(\frac{5\pi}{2} - \Theta\right)}{2\pi(\pi^2 + 2\pi\Theta - \Theta^2)}. \tag{45}$$

Equations (44) and (45) lead to results that are very close to each other. However, strictly speaking, the results are the same only for $\Theta = 0, \pi$ and 2π (respectively, $\langle u^2\rangle = 0, 3/8$

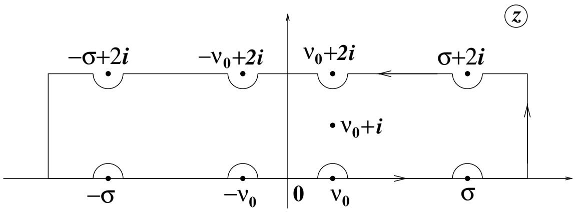

Figure 4. The contour used to compute $P(\phi)$, equations (48) and (50).

and 1). We can also note that for small Θ values, the behaviour at lowest order are different ($\langle u^2\rangle = \frac{5}{4\pi^2}\Theta^2 + \cdots$ with equation (45) and $\langle u^2\rangle = \left(\frac{3}{8\pi^2} + \frac{3}{32}\right)\Theta^2 + \cdots$ with equation (44)).

After the computation of the third moment, we will come back to this discussion and consider, in particular, the case of a wedge angle close to π.

In order to compute $\langle u^3\rangle$, we write (32) as

$$2a_3(\sigma) = \frac{\sigma^2}{4}a_1(\sigma) - \sigma\sinh(\sigma(\pi-\alpha))M(\alpha) - \cosh(\sigma(\pi-\alpha))\frac{\mathrm{d}M}{\mathrm{d}\psi}(\psi=\alpha) \tag{46}$$

$$M(\psi) = \frac{1}{2}\left(1 - \frac{\sigma^2}{4}\right)K(\psi) + \frac{1}{8}K''(\psi) + \int_0^\infty \mathrm{d}\nu\,\frac{\cosh(\nu\psi)a_2(\nu)}{\cosh(\pi\nu) - \cosh(\pi\sigma)}, \tag{47}$$

with $a_2(\sigma)$ taken in (43).

The computation of the last integral in (47) is very long and involves the following integral, $P(\phi)$, and its derivatives:

$$P(\phi) = \lim_{\nu_0\to 0}\int_{-\infty}^{\infty}\mathrm{d}\nu\,\frac{\cosh(\nu\phi)}{\sinh((\nu+\nu_0)\pi/2)\sinh((\nu-\nu_0)\pi)(\cosh(\nu\pi)-\cosh(\sigma\pi))} \tag{48}$$

$$= \frac{1}{\cos\phi}\left[\frac{1}{1+\cosh(\pi\sigma)} + \frac{2\sin\phi\sinh(\sigma\phi)}{\sinh(\pi\sigma/2)(\sinh(\pi\sigma))^2} + \frac{2\phi\sin\phi}{\pi(1-\cosh(\pi\sigma))}\right] \tag{49}$$

$$= \frac{1}{2\cos\phi} + 2\tan\phi\left(\frac{1}{3}\left(\frac{\phi}{\pi}\right)^3 - \frac{7}{12}\left(\frac{\phi}{\pi}\right)\right) \quad \text{in the limit } \sigma\to 0. \tag{50}$$

(The result (49) is again obtained by contour integration—see figure 4 for the contour.)

Finally, we get, for a general wedge of magnitude Θ, the following complicated expression of the third moment $\langle u^3\rangle$, $\left(X \equiv \frac{\Theta}{\pi}, s \equiv s(X) = \sin(\pi X), c \equiv c(X) = \cos(\pi X),\right.$ $\left. t \equiv t(X) = \tan(\pi X)\right)$:

$$\langle u^3\rangle_{(\Theta)} = \frac{3}{2}\langle u^2\rangle - \frac{1}{2}\langle u\rangle + \frac{1}{64}N_3$$

$$N_3 = \frac{10}{\pi^2} - \frac{4}{\pi^2}X - \frac{6}{c^2}\left[8(\langle u^2\rangle - \langle u\rangle) - \frac{2}{3}X^2 + \frac{4}{3}X + \frac{1}{\pi^2} + \frac{1}{4}\right]$$
$$+\left(\frac{1}{c^2} + \frac{2}{s^2}\right)A + \left(t + \frac{2}{sc}\right)B + (2X+1)\frac{2t^2}{\pi^2} + \frac{2s+3s^3}{4c^4}$$
$$+t\left(\frac{5Rs}{c^3} + (5t^2+4)D - \frac{1}{2} + \frac{13}{9\pi} + \frac{4}{3\pi^3}\right)$$
$$+\left(1 + \frac{1}{c^2}\right)W(X) - \frac{1}{c^2}W(2X)$$

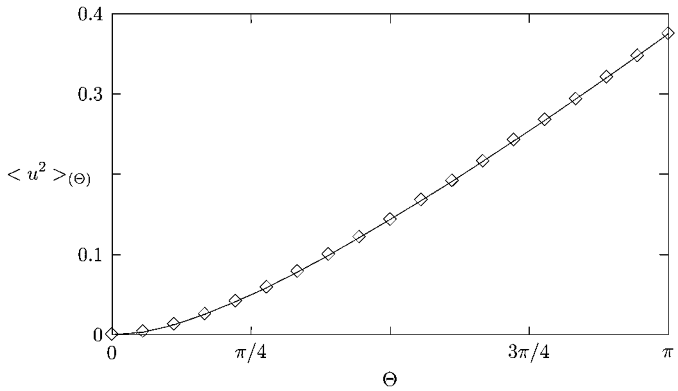

Figure 5. The second moment $\langle u^2 \rangle$ of a wedge occupation time as a function of the wedge angle. The line is equation (44). The points are numerical simulations. For further explanations, see text.

Table 1. Analytical values of $\langle u^2 \rangle$ and $\langle u^3 \rangle$ for some values of the wedge angle Θ.

Θ	$\langle u^2 \rangle_{(\Theta)}$	$\langle u^3 \rangle_{(\Theta)}$
$\frac{\pi}{4}$	$\frac{7}{64} - \frac{3}{16\pi} = 0.049\,69\ldots$	$\frac{21}{512} + \frac{\sqrt{2}}{64} - \frac{151}{1152\pi} + \frac{3}{64\pi^2} + \frac{1}{48\pi^3} = 0.026\,81\ldots$
$\frac{\pi}{3}$	$\frac{49}{288} - \frac{\sqrt{3}}{6\pi} = 0.078\,25\ldots$	$-\frac{1435}{15\,552} + \frac{15\sqrt{3}}{128} - \frac{349\sqrt{3}}{3456\pi} - \frac{1}{12\pi^2} + \frac{\sqrt{3}}{48\pi^3} = 0.047\,74\ldots$
$\frac{\pi}{2}$	$\frac{5}{32} - \frac{1}{8\pi^2} = 0.143\,58\ldots$	$\frac{63}{512} - \frac{7}{32\pi^2} = 0.100\,88\ldots$
$\frac{2\pi}{3}$	$\frac{49}{288} + \frac{\sqrt{3}}{12\pi} = 0.216\,08\ldots$	$\frac{715}{15\,552} + \frac{9\sqrt{3}}{128} + \frac{13\sqrt{3}}{3456\pi} - \frac{1}{24\pi^2} - \frac{\sqrt{3}}{48\pi^3} = 0.164\,44\ldots$
$\frac{3\pi}{4}$	$\frac{15}{64} + \frac{1}{16\pi} = 0.254\,26\ldots$	$\frac{79}{512} + \frac{\sqrt{2}}{64} + \frac{79}{1152\pi} + \frac{1}{64\pi^2} - \frac{1}{48\pi^3} = 0.199\,13\ldots$
π	$\frac{3}{8} = 0.375$	$\frac{5}{16} = 0.3125$

$$A = -\frac{2}{3}X(X-1)(X-2); \qquad B = -\frac{1}{\pi}A' \qquad \left(A' \equiv \frac{\mathrm{d}A}{\mathrm{d}X}\right)$$

$$R = \frac{1}{3}X\left(X^2 - \frac{7}{4}\right); \qquad D = \frac{1}{\pi}R'$$

$$W(X) = \frac{1}{c(X)^2}\left(-\frac{s(X)}{4} + E(X)\right) - t(X)F(X)$$

$$E(X) = -\frac{1}{3}\left(X^3 - 6X^2 + \frac{41}{4}X - \frac{9}{2}\right); \qquad F(X) = -\frac{1}{\pi}E'. \tag{51}$$

The values of $\langle u^2 \rangle$ and $\langle u^3 \rangle$ for some special values of Θ are quoted in table 1.

In figures 5 and 6, we have plotted $\langle u^2 \rangle$ and $\langle u^3 \rangle$ as functions of the wedge angle Θ. The full lines represent, respectively, equations (44) and (51). The points come from numerical simulations (100 000 random walks with 100 000 steps for each one; to avoid numerical artefacts with the boundaries, we did not use any lattice). The agreement between simulations and theory is better than 0.8% for $\langle u^2 \rangle$ and $\langle u^3 \rangle$ and for all the Θ range.

It is worth noting that our results are very close to those obtained with the beta law (7). For instance, for $\Theta = \pi/3\,(2\pi/3)$, (8) should give, respectively, $\langle u^2 \rangle = 13/268 = 0.077\,38\ldots\ (11/51 = 0.215\,68\ldots)$ and $\langle u^3 \rangle = 299/6384 = 0.046\,83\ldots\ (209/1275 = 0.163\,92\ldots)$. As another example, let us briefly discuss the case $\Theta = \pi + \epsilon$ ($|\epsilon| \ll 1$). With

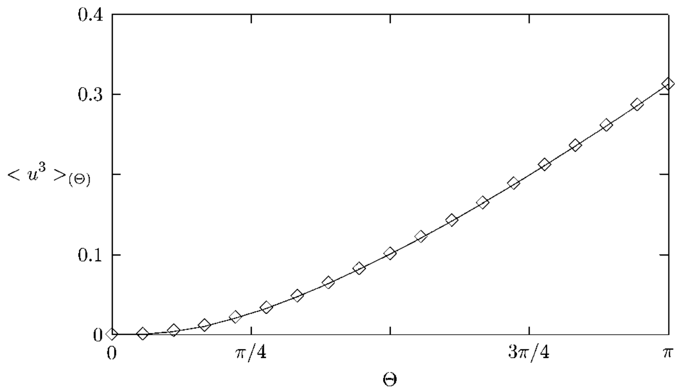

Figure 6. The same as figure 5 but for the third moment $\langle u^3 \rangle$. The line is equation (51).

equations (44) and (51), we get

$$\langle u^2 \rangle = \frac{3}{8} + \frac{\epsilon}{2\pi} + \cdots \tag{52}$$

$$\langle u^3 \rangle = \frac{5}{16} + \frac{23\epsilon}{48\pi} + \cdots. \tag{53}$$

At order ϵ, we obtain the same results with equation (8). (However, the ϵ^2 terms would be different.) This illustrates again that the distribution of u is quite close to (7).

For the quadrant $\left(\Theta = \frac{\pi}{2}\right)$, we observe that our result

$$\langle u^3 \rangle_{(\pi/2)} = \frac{63}{512} - \frac{7}{32\pi^2} \tag{54}$$

disagrees with the one of [5] (see equation 5) of the present paper). However, in [5], p 130, (ix), the authors quote the result $\frac{1}{8\pi^2}\int_0^1\int_0^1 \mathrm{d}u\,\mathrm{d}v\, uv^3 \arcsin u \arcsin uv = \frac{7}{144\pi} - \frac{5}{36\pi^2}$ that seems to be incorrect and should be replaced by $\frac{5}{2048} - \frac{1}{64\pi^2}$. Indeed, with this change in equation (5), we recover result (54).

Looking at table 1, we remark that the quadrant problem seems to be the simplest. This is consistent with the conjecture of Meyre and Werner—see equations (6) and (9) of the present paper.

In particular, the moments $\langle u^2 \rangle$ and $\langle u^3 \rangle$ take the same form $q_1 + \frac{q_2}{\pi^2}$, where q_1 and q_2 are rational numbers. Thus, a natural question arises: is $\langle u^4 \rangle$ still of the same form or even close to it, for instance with an additional $\frac{1}{\pi^4}$ term only? Clearly, we even expect that the knowledge of $\langle u^4 \rangle$ could give some interesting information concerning the probability density of u.

After a lot of contour integrations (think, for instance, of the integral involving $a_3(\sigma)$) and a huge amount of computation (finished with the help of Maple), we get the result

$$\langle u^4 \rangle_{(\pi/2)} = \frac{907}{8192} - \frac{1}{64\pi} - \frac{631}{2304\pi^2} - \frac{7}{384\pi^4} = 0.0778\ldots \tag{55}$$

to be compared with numerical simulations (0.0772—same conditions as for figures 5 and 6). Remark that the beta law (7) should lead to $\langle u^4 \rangle = \frac{1}{13} = 0.0769\ldots$.

Of course, result (55) is disappointing, especially because of the presence of the $\frac{1}{\pi}$ term. Thus, even for the quadrant problem, things seem to be quite complicate. In principle, following the same lines, it could be possible to compute $\langle u^5 \rangle_{(\pi/2)}$. However, the expected computational volume of, among others, the integral involving $a_4(\sigma)$ seems prohibitive. Thus, we stopped here our study of the planar BM.

3. Orthants occupation times

Let us now turn to the n-dimensional case and study a BM starting, as before, from O at $t = 0$ and stopped at t somewhere in $\mathbf{R}^n$. T is, now, the time spent in the orthant $\{x_1 > 0, x_2 > 0, \ldots, x_n > 0\}$ ($\equiv \mathbf{O}_n$) and $u = \frac{T}{t}$ the corresponding reduced variable.

For the computation of the second moment, we will use the standard perturbation theory. Starting with equations (12) and (13) where, this time, $d \equiv \mathbf{O}_n$, and expanding in p, we have, for the p^2 term,

$$\frac{1}{2}\langle T^2\rangle = \int_0^t \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_1 \int_{\mathbf{R}^n} \mathrm{d}\overrightarrow{R} \int_{\mathbf{R}^n} \mathrm{d}\vec{r} \int_{\mathbf{R}^n} \mathrm{d}\vec{r}\,' \mathbf{1}_{\mathbf{O}_n}(\vec{r})\mathcal{G}(\vec{r}; t_1) \times \mathbf{1}_{\mathbf{O}_n}(\vec{r}\,')\mathcal{G}(\vec{r}\,' - \vec{r}; t_2 - t_1)\mathcal{G}(\overrightarrow{R} - \vec{r}\,'; t - t_2) \tag{56}$$

$$= \int_0^t \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_1 \int_{\mathbf{O}_n} \mathrm{d}\vec{r} \int_{\mathbf{O}_n} \mathrm{d}\vec{r}\,' \mathcal{G}(\vec{r}; t_1)\mathcal{G}(\vec{r}\,' - \vec{r}; t_2 - t_1), \tag{57}$$

where $\mathcal{G}$ is the n-dimensional free propagator satisfying the equation

$$(\partial_t - D\Delta)\mathcal{G}(\vec{r}; t) = \delta(\vec{r})\delta(t) \tag{58}$$

So

$$\mathcal{G}(\vec{r}; t) \equiv \mathcal{G}(x_1, \ldots, x_n; t) = \left(\frac{1}{4\pi Dt}\right)^{n/2} \exp\left(-\frac{r^2}{4Dt}\right) \tag{59}$$

$$= \prod_{j=1}^{n} G(x_j; t). \tag{60}$$

Here $x_1, \ldots, x_n$ are the coordinates of $\vec{r}$ and G is the one-dimensional free propagator:

$$G(x; t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\right).$$

Rescaling the time in (57) and introducing the reduced variable u, we readily get

$$\frac{1}{2}\langle u^2\rangle = \int_0^1 \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_1 \int_{\mathbf{O}_n}\int_{\mathbf{O}_n} \mathrm{d}\vec{r}\,\mathrm{d}\vec{r}\,'\,\mathcal{G}(\vec{r}; t_1) \cdot \mathcal{G}(\vec{r}\,' - \vec{r}; t_2 - t_1) \tag{61}$$

$$= \int_0^1 \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_1 \left[\int_0^\infty \int_0^\infty \mathrm{d}x\,\mathrm{d}x'\,G(x; t_1) \cdot G(x' - x; t_2 - t_1)\right]^n. \tag{62}$$

Now, introducing the variable $t' = \frac{t_1}{t_2}$, we get, after some simple manipulations,

$$\langle u^2\rangle = \frac{1}{(2\pi)^n} \int_0^1 \mathrm{d}t' \left[\frac{1}{2D\sqrt{t'(1-t')}} J\right]^n \tag{63}$$

$$J = \int_0^\infty \int_0^\infty \mathrm{d}x\,\mathrm{d}x' \exp\left(-\frac{(x'-x)^2}{4D(1-t')}\right) \cdot \exp\left(-\frac{x^2}{4Dt'}\right) = 2D\sqrt{t'(1-t')}\left[\frac{\pi}{2} + \arctan\sqrt{\frac{t'}{1-t'}}\right]. \tag{64}$$

As expected, the diffusion constant D cancels in equation (63) and we are left with

$$\langle u^2\rangle = \frac{1}{(2\pi)^n} \int_0^1 \mathrm{d}t' \left(\frac{\pi}{2} + \arctan\sqrt{\frac{t'}{1-t'}}\right)^n = \frac{1}{2^{2n+1}\pi^n} \int_0^\pi \mathrm{d}\phi\,\sin\phi\,(\pi + \phi)^n. \tag{65}$$

Table 2. Analytical values of $\langle u^2\rangle$, equation (66), for some values of the dimension n.

n	$\langle u^2\rangle$
1	$\frac{3}{8}$
2	$\frac{5}{32} - \frac{1}{8\pi^2}$
3	$\frac{1}{128}\left(9 - \frac{18}{\pi^2}\right)$
4	$\frac{1}{512}\left(17 - \frac{60}{\pi^2} + \frac{48}{\pi^4}\right)$
5	$\frac{1}{2048}\left(33 - \frac{180}{\pi^2} + \frac{360}{\pi^4}\right)$
6	$\frac{1}{8192}\left(65 - \frac{510}{\pi^2} + \frac{1800}{\pi^4} - \frac{1440}{\pi^6}\right)$

Finally, we obtain [10] the general formula for the second moment of an orthant occupation time

$$\langle u^2\rangle = \frac{1}{2^{2n+1}}\left(\sum_{k=0,2,4,\ldots}^{n} k!\binom{n}{k}\pi^{-k}(2^{n-k}+1)\cos\frac{k\pi}{2}\right). \tag{66}$$

The values of $\langle u^2\rangle$ for the lowest dimensions are quoted in table 2.

$n = 1$ is Levy's arc-sine law. For $n = 2$, we recover $\langle u^2\rangle_{(\pi/2)}$ of the previous section. For $n = 3$, our result $\frac{1}{128}\left(9 - \frac{18}{\pi^2}\right) = 0.056\,06\ldots$, that is in agreement with numerical simulations (0.056 28), disagrees with the one of [5]: $\frac{9}{128} - \frac{3}{32\pi^2} - \frac{3}{64\pi^3} = 0.059\,30\ldots$. Because of the lack of computational details, we can not discuss this discrepancy (maybe a simple misprint—think of the change of $\frac{3}{64\pi^3}$ into $\frac{3}{64\pi^2}$ in the result of [5]).

Now, if we want to go to higher moments of u, following (62), we have to perform the spatial integration of a product of at least three free propagators. We did not succeed to get a result in a closed form. Thus, to our knowledge, this problem is still an open one.

4. Summary

We have developed a method to compute the moments of a wedge occupation time for planar BM. It allows us to get explicitly the first moments for any wedge angle. However, the application of this method is limited because of exponentially growing calculations. Actually, it is difficult to go beyond the fourth moment. Moreover, in the quadrant case (apparently, the simplest one), it does not bring clear information on the occupation time density.

Concerning the orthants, despite the fact that we could get a general formula for the second moment, we also rapidly met serious technical difficulties.

Finally, our feeling is that the occupation time problem for BM is far from being understood as soon as we leave one-dimensional or quasi-one-dimensional (graphs) situations. Clearly, new ideas are needed if we want to tackle this problem.

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